Green's Theorem

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A path/curve (called C or γ) starts at some $\vec{a}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and end at some $\vec{b}$.

→ If $\vec{a} = \vec{b}$ it's closed, so you must specify the direction.

→ Reversing the direction sends C to $-C$ and:

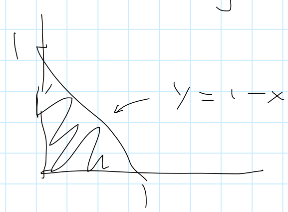
$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot dr$$

$$\int_{-C} F\,ds = \int_C F\,ds$$

$$\boxed{1 - x - y = 0}$$

$x + y + \cancel{z} = 1$

$\left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\}$



FTC for line integrals

Suppose C is a path/curve from $\vec{a}$ to $\vec{b}$, and F is a scalar function defined on an open domain $D \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$) containing C.

Let $\vec{F} = \nabla F$.

Then

$$\int_C \vec{F} \cdot d\vec{r} = F(\vec{b}) - F(\vec{a})$$

Remarks

① $\int_{-C} = -\int_C$ makes sense in terms of the FTC

bc if C is from $\vec{a}$ to $\vec{b}$, then $-C$ is from $\vec{b}$ to $\vec{a}$ and $F(\vec{a}) - F(\vec{b}) = -(F(\vec{b}) - F(\vec{a}))$

② FTC says $\int_C \vec{F} \cdot d\vec{r}$ only depends on the endpoints of C & not on the particular path btwn them if $\vec{F} = \nabla F$.

BUT for many $\vec{F}$, the integral does depend on the path

⇒ $\vec{F}$ is not a gradient of some F.

(converse is "basically" true)

Defn If $\vec{F} = \nabla F$, Then F is called a potential for

Conservative ⇒ $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints

Defn $\vec{F}$ is conservative if $\vec{F}$ has a potential.

③ If $\vec{F}$ is conservative, then $\int_C \vec{F} \cdot d\vec{r} = 0$ if C is closed.

(bc $F(\vec{a}) - F(\vec{a}) = 0$)

Recall If $(x(t), y(t))$ is a parameterization of a curve C for $a \le t \le b$, then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

$$= \int_a^b \left[P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)\right] dt$$

* similar for $\mathbb{R}^3$ but w/ z also.

independent of parameterization

Proof of FTC (in $\mathbb{R}^2$)

suppose $\vec{F} = \nabla F$ and choose a parameterization $(x(t), y(t))$ for $a \le t \le b$.

Proof of FTC (in $\mathbb{R}^2$)

suppose $\vec{f} = \nabla F$ and choose a parameterization $(x(t), y(t))$ for $a \le t \le b$.

Let $\vec{a}, \vec{b}$ be the endpoints of C.

$\Rightarrow$ $\vec{a} = (x(a), y(a)) = g(a)$

$\vec{b} = (x(b), y(b)) = g(b)$

$$\int_C \nabla F \cdot d\vec{r} = \int_C \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$

$$= \int_{t=a}^{t=b} \underbrace{\left[\frac{\partial F}{\partial x}(x(t), y(t)) x'(t) + \frac{\partial F}{\partial y}(x(t), y(t)) y'(t)\right]}_{\text{use MV chain rule to rewrite}} dt$$

$g : [a,b] \to \mathbb{R}^2$ $\quad g(t) = (x(t), y(t))$

so the derivative is $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

$F : \mathbb{R}^2 \to \mathbb{R}$

so the derivative is $\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}$

so the derivative of $F \circ g : [a,b] \to \mathbb{R}$ is

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \frac{\partial F}{\partial x} x'(t) + \frac{\partial F}{\partial y} y'(t)$$

= the integrand we had above

$= \frac{d}{dt}(F \circ g)$

$$\Rightarrow \quad = \int_{t=a}^{t=b} \left[\frac{\partial F}{\partial x}(x(t), y(t)) x'(t) + \frac{\partial F}{\partial y}(x(t), y(t)) y'(t)\right] dt$$

$$= \int_a^b \frac{d}{dt}(F \circ g)\, dt$$

$\underset{\text{by SV FTC}}{=} \left[F \circ g\right]_a^b$

$= F(g(b)) - F(g(a))$

$= F(\vec{b}) - F(\vec{a})$

$\blacksquare$ QED

Suppose C_1 is a curve from $\vec{a}$ to $\vec{b}$ and C_2 is from $\vec{b}$ to $\vec{c}$, then get $C_1 + C_2$ from $\vec{a}$ to $\vec{c}$ "go along C_1, then along C_2"

Key fact

$$\int_{C_1 + C_2} \vec{f} \cdot d\vec{r} = \int_{C_1} \vec{f} \cdot d\vec{r} + \int_{C_2} \vec{f} \cdot d\vec{r}$$

Compare: $\displaystyle\int_a^b f\,dx + \int_b^c f\,dy = \int_a^c f\,dx$

Suppose we're given parameterization g_1 of C_1 and g_2 of C_2, each defined from $0 \le t \le 1$.

Q/ How to write parameterization g of $C_1 + C_2$ s.t. g is defined for $0 \le t \le 1$?

Q/ Given $t \in [0,1]$, what is $g(t)$?

A/

$$g(t) = \begin{cases} g_1(2t) & 0 \le t \le \frac{1}{2} \\ g_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

At $t = \frac{1}{2}$, the def is consistent because we assumed that the endpoint of C (aka $g_1(1)$) is the initial point of C_2 (aka $g_2(0)$)

Technical point

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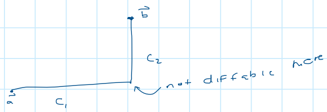
Technical point

When we compute line integrals using a param $(x(t), y(t))$
we take derivatives of the components. This requires
that they are diff'able fcns of t.

<u>But</u> what if we have a curve C like this?



A/ We can still compute $\int_C$ by writing $C = C_1 + C_2$
and then $\int_C = \int_{C_1} + \int_{C_2}$ if C is piecewise
smooth but not smooth

<u>Note</u> $C_1 + C_2 = C_1 \cup C_2$

Compatibility btwn adding curves and FTC:

For $\vec{F} = \nabla F$, the fact that

$$\int_{C_1 + C_2} = \int_{C_1} + \int_{C_2}$$

is equivalent to:

$$F(\vec{c}) - F(\vec{a}) = (F(\vec{c}) - F(\vec{b})) + (F(\vec{b}) - F(\vec{a}))$$

<u>Recall</u>

if $\vec{F}$ is conservative then

② $\int_C \vec{F} \cdot d\vec{r}$ is path-independent

③ $\oint \vec{F} \cdot d\vec{r} = 0$ if C is closed

② $\Leftrightarrow$ ③ for any $\vec{F}$

<u>② $\Rightarrow$ ③</u> If C (from $\vec{a}$ to $\vec{a}$) is closed, let C' be
the const path from $\vec{a}$ to $\vec{a}$
(ie C' parameterized by $g(t) = \vec{a}$)
then $g'(t) = 0$

$$\Rightarrow \int_{C'} \vec{F} \cdot d\vec{r} = \int \vec{F}(g(t)) \cdot g'(t)\, dt = \int 0\, dt = 0$$

<u>but</u> let C and C' have same endpoint

$$\Rightarrow \text{by ② } \int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r} = 0$$

<u>③ $\Rightarrow$ ②</u> Suppose ③ is true and that C and C' have the
same endpoints $\vec{a}$ and $\vec{b}$

<u>Consider</u> $-C'$ which is from $\vec{b}$ to $\vec{a}$ and

$$C'' = C - C'$$
$$= C + (-C')$$

= "go along C, then go along C' in the other direction"

$\Rightarrow C''$ is closed

by ③, $\int_{C''} \vec{F} \cdot d\vec{r} = 0$

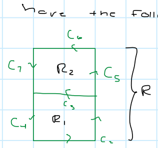
But $\int_{C''} \vec{F} \cdot d\vec{r} = \int_{C + (-C')} \vec{F} \cdot d\vec{r}$

$$= \int_C \vec{F} \cdot d\vec{r} + \int_{-C'} \vec{F}\, d\vec{r} = \int_C \vec{F} \cdot d\vec{r} - \int_{C'} \vec{F}\, d\vec{r}$$

$$\Rightarrow \int_C \vec{F}\, d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r}$$

■ QED Thm 43

Suppose we have the following curves.



$R_1 \to$ bottom □
$R_2 \to$ top □
$R \to$ whole □

Consider some vector field $\vec{F} = (P, Q)$

$$\int_{R_1} \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2 + C_3 + C_4} \vec{F} \cdot d\vec{r}$$

ccw by default

$$\int_{R_2} \vec{F} \cdot d\vec{r} = \int_{C_5 + C_6 + C_7 - C_3} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_{R_1} \vec{F} \cdot d\vec{r} + \int_{R_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1 + C_2 + \cancel{C_3} + C_4} \vec{F} \cdot d\vec{r} + \int_{C_5 + C_6 + C_7 - \cancel{C_3}} \vec{F} \cdot d\vec{r} = \int_{R} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1 + C_2 + C_4 + C_5 + C_6 + C_7} \vec{F} \cdot d\vec{r}$$

but going around ccw is $C_1 + C_2 + C_5 + C_6 + C_7 + C_4$